# Reply to "Comment on 'Nonstationary optimal paths and tails of prehistory probability density in multistable stochastic systems'"

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We argue that the validity of the WKB approach for calculating nonequilibrium properties of a nonlinear stochastic system requires an additional criterion involving the time interval, compared with the standard WKB criterion considered by Mannella [R. Mannella, Phys. Rev. E **58**, 2479 (1999)]. In order to clarify the situation, we compare the WKB solution with the direct numerical solution of the Fokker-Plank equation.
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In Ref. [1] we applied the WKB approximation in order to calculate nonequilibrium properties of nonlinear stochastic systems such as the transition probability $P(x_f,t_f;x_0,t_0)$ to find the system at point $x_f$ at time $t_f$ if at time $t_0$ it was located at $x_0$. The notion of a nonstationary optimal path was introduced as a trajectory of an auxiliary mechanical system corresponding to minimum action between points $(x_0,t_0)$ and $(x_f,t_f)$. Mannella's analysis [2] suggests that for nonstationary optimal trajectories the situation can be more complicated than for stationary trajectories due to the existence of multiple solutions.

For an overdamped one-dimensional system with potential $U(x)=-\frac{1}{2}x^2+\frac{1}{4}x^4$ and some particular values of the initial and final point and long enough time interval (e.g., for $x_0=-0.5$, $x_f=-0.1$, and $t\equiv t_f-t_0=8$ chosen in Ref. [1]), two nonstationary optimal trajectories were found [2] and it was shown that the minimum action $S=0.22$ corresponds to the trajectory which moves first to the unstable point $x=0$, stays there for some time, and then moves back to the final point [3]. The other solution (with $S=0.49$), presented in Ref. [1], corresponds instead to the trajectory which first goes to the attractor $x=-1$ and stays there before reaching the final point (note that only this type of solution is realized in linear systems). Mannella proposed that the condition of minimal action might be the natural criterion for choosing one or another trajectory in analogy with stationary optimal trajectories.

We emphasize, however, that for nonstationary trajectories the criterion of minimal action has only a limited range of applicability and depends on the duration of the time interval $t$. Indeed, according to the general properties of stochastic processes, the system under consideration should possess the characteristic time $\tau_D$ for which the system approaches its quasiequilibrium state in the basin of a given attractor, described by the probability density $P^{\mathrm{eq}}(x_f)$. That is, $P(x_f,t_f;x_0,t_0)\to P^{\mathrm{eq}}(x_f)$ at $t\gg\tau_D$, where $P(x_f,t_f;x_0,t_0)\propto\exp(-S(x_f,t_f;x_0,t_0)/D)$, $P(x_f)\propto\exp(-S^{\mathrm{eq}}(x_f)/D)$, $S^{\mathrm{eq}}(x_f)=2[U(x_f)-U(x_0)]$, and $D$ is the diffusion coefficient. Since $S^{\mathrm{eq}}(-0.1)=0.49$, the criterion of minimum action in choosing the nonstationary optimal path in the presence of multiple solutions contradicts the tendency of approaching the quasiequilibrium state. In fact, if the time interval $t$ is long enough, in order to satisfy the condition of minimal action the system has to be located in a nonequilibrium state near the unstable point for a long period of time. This difficulty does not arise for the solution chosen in Ref. [1], which reflects the tendency of approaching the equilibrium.

On the other hand, one can expect that the criterion of minimal action would select the true optimal trajectory for $t\ll\tau_D$. Since $\tau_D$ depends on the value of the diffusion coefficient, the choice in favor of one or the other optimal trajectory cannot be made *a priori* based on the WKB approximation. The direct solution of the Fokker-Plank (FP) equations is required.

In order to clarify the situation, we have solved the FP equation for the transition probability corresponding to the stochastic Langevin equation studied in Refs. [1,2]. Based on the solution obtained, we present in Fig. 1 the calculated values of $-D\ln P(-0.1,t;-0.5,0)$ and compare them with the values of $S(-0.1,t;-0.5,0)$ for the two optimal solutions in the WKB approach [2]. One can see that the optimal solution corresponding to the steady-state value $S=0.49$ appears for $|t|\gtrsim 6$. This optimal solution is close to the solution

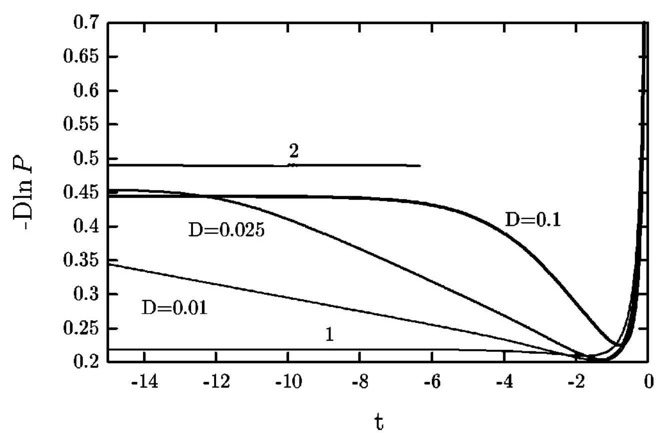


FIG. 1. The values of $-D\ln P(-0.1,0;-0.5,0)$ for $D=0.1$, 0.025, and 0.001 obtained by the numerical solution of the Fokker-Plank equation. Curves 1 and 2 represent the values of $-D\ln P(-0.1,0;-0.5,0)$ obtained with the use of the WKB approximation, which show the existence of multiple solutions for a long time interval.

 

given by the FP equation for the values of $D \sim 0.1$ (the deviation of the values $-D \ln P$ from their optimal values is caused by the preexponential factor, not included in the WKB solution). At the same time, for smaller values of $D$ more time is required for the system to approach the quasiequilibrium. However, one can see from Fig. 1 that the optimal solution with minimum action would adequately describe the transition probability only for extremely small values of $D$, leading to vanishing values of the transition probability which are outside of the typical experimentally accessible regime.

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[1] B. E. Vugmeister, J. Botina, and H. Rabitz, Phys. Rev. E **55**, 5338 (1997).

[2] R. Mannella, Phys. Rev. E **59**, 2479 (1999).

[3] Note that in order to be consistent with the concept of optimal fluctuations, assuming that the normalization coefficient (prefactor) of the probability density is small within logarithmic accuracy, one needs to add the constant 1/4 to the potential $U(x)$, leading to zero energy of the system in the ground state (attractor).